Photonic Floquet Topological Insulator in an Atomic Ensemble

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We demonstrate the photonic Floquet topological insulator (PFTI) in an atomic vapor with nonlinear susceptibilities. The interference of three coupling fields splits the energy levels periodically to form a periodic refractive index structure with honeycomb symmetry that can be adjusted by the choice of frequency detunings and intensities of the coupling fields, which all affect the appearance of Dirac cones in the momentum space. When the honeycomb lattice sites are helically ordered along the propagation direction, we obtain a PFTI in the atomic vapor in which an obliquely incident beam moves along the zigzag edge without scattering energy into the PFTI, due to the confinement of the edge states. The appearance of Dirac cones and the formation of PFTI is strongly affected by the nonlinear susceptibilities; i.e. the PFTI can be shut off by the third-order nonlinear susceptibility and re-opened up by the fifth-order one.
Recently, topological insulators (TIs) have attracted much attention. A TI, as a new phase of matter, allows free electrons to exist only on the surface, so that the moving electrons – the surface currents – will not be affected by defects or disorders\(^1\text{-}\text{3}\). There are conducting edge states of a TI that lie in the bulk energy gap in the momentum space and are spatially localized on the boundaries of TI. These edge states are predicted to be useful in performing quantum computations\(^4\). TIs, as well as some graphene-based structures, have also found potential applications in optical modulators\(^5\text{-}\text{6}\) and optical diodes\(^7\). Photonic topological insulators (PTIs), fabricated by utilizing metamaterials\(^8\) or helical waveguides\(^9\), can break the time-reversal symmetry and lead to a one-way edge state, which is robust against defects. So far, research on PTIs is mostly based on graphene-like structures. A honeycomb lattice\(^10\text{-}\text{14}\) also exhibits graphene-like properties and can be generated by the femtosecond laser writing technique or the three-beam interference method\(^15\text{-}\text{16}\). The first method is valid only in solid materials, whereas the second method can be used in both solid and gas materials\(^17\text{-}\text{19}\). We note that three-beam interference will induce a hexagonal lattice instead of honeycomb lattice generally. However, the corresponding refractive index change will exhibit honeycomb profile for a saturable nonlinear medium or in atomic vapors.

The interference pattern (hexagonal lattice) produced by the three-beam interference exhibits many pairs of singularities, and the band structure of the corresponding refractive index change (honeycomb lattice) features conical singularities at the corners of the first Brillouin zone. In an atomic (such as rubidium) vapor, when the three-beam interference pattern serves as the dressing field, the dressed atomic system will exhibit controllable optical properties, which were extensively investigated\(^20\text{-}\text{22}\) in the past decade. In this article, we demonstrate the creation of a photonic Floquet topological insulator (PFTI) in an atomic ensemble, which, to the best of our knowledge, has never been reported before. Even though there is a related work done in ultracold fermionic atoms\(^23\), the topic in this work is quite different. One of the main advantages of generating PFTIs in atomic vapors is that many interesting topological properties can be easily controlled through adjusting frequency detunings and power of the coupling fields, as well as high-order nonlinear susceptibilities, in multi-level atomic systems.

**Results**

**System preparation.** We consider an inverted Y-type atomic system with electromagnetically induced transparency (EIT), as shown in Fig. 1(a). In the system the field \(E_p\) probes the transition \(|0\rangle \rightarrow |1\rangle\), the coupling field \(E_2\) drives the transition \(|1\rangle \rightarrow |2\rangle\), and the controlling field \(E_3\) connects \(|1\rangle \rightarrow |3\rangle\). The proposed scheme is appropriate for an atomic system in a regular cold magneto-optical trap. If the three coupling fields of the same frequency are launched paraxially along the propagation direction \(z\), they will interfere with each other\(^24\text{-}\text{26}\) to form a two-dimensional hexagonal interference pattern in the transverse \(xy\) plane. The resulting Rabi frequency of such optically induced interference pattern can be written as
where $\theta_i = [0, 2\pi/3, 4\pi/3]$ are the relative angles between the three laser beams, $k_2$ is the wavenumber of the coupling fields, $G_2$ represents the Rabi frequency of the coupling fields with $G_2 = \varphi_{12}E_2/h$, where $\varphi_{12}$ is the electric dipole moment. Level $|1\rangle$ can be dressed by the coupling fields and will split into two sublevels $|\pm\rangle$ having modified eigenfrequencies $\xi_{\pm} = -\Delta_2/2 \pm \sqrt{\Delta_2^2/4 + |G|^2}$ with $|G|^2 = |G_2|^2[4 \cos(3k_2x/2) \cos(\sqrt{3}k_2y/2) + 2 \cos(\sqrt{3}k_2y) + 3]$. Since the three coupling fields interfere with each other to form the periodic interference pattern, the sublevels $|\pm\rangle$ will be periodic, as shown in Fig. 1(b) and the inset panels, in which the grid represents the original level $|1\rangle$.

Since the energy level is dressed to be periodic, the susceptibility will also have the same periodicity, and can be written as

$$\chi_p(x, y) = \chi_p^{(1)} + \chi_p^{(3)}|E_2|^2 + \chi_p^{(5)}|E_2|^4,$$

where only the first three (linear and nonlinear) susceptibilities are considered. In Eq. (2),

$$\chi_p^{(1)} = iN\varphi_{10}^2/\{\hbar\epsilon_0[(d_{10} + |G|^2/d_{20})]\}, \quad \chi_p^{(3)}|E_2|^2 = -iN\varphi_{10}^2G_2^2/\{\hbar\epsilon_0[d_{20}(d_{10} + |G|^2/d_{20})]^2\},$$

and

$$\chi_p^{(5)}|E_2|^4 = iN\varphi_{10}^2G_2^2/\{\hbar\epsilon_0[d_{20}(d_{10} + |G|^2/d_{20})]^3\}$$

with $N$ being the atomic density, $\varphi_{10}$ the electric dipole moment, and $d_{10} = \Gamma_{10} + i\Delta_1$, $d_{20} = \Gamma_{20} + i(\Delta_1 - \Delta_2)$. Here $\Gamma_{ij}$ are the decay rates between $|i\rangle$ and $|j\rangle$, $\Delta_1 = \Omega_{10} - \omega_p$ and $\Delta_2 = \Omega_{12} - \omega_2$ are the frequency detunings, $\Omega_{ij}$ are the transition frequencies between $|i\rangle$ and $|j\rangle$, and $\omega_p$ ($\omega_2$) is the frequency of the probe (coupling) field. We would like to note that the optical lattice structures in atomic systems are quite stable, as long as the laser beams forming the optical lattice are stable; small fluctuations should not affect our main results.

**Band structure.** In Figs. 2(a1)-2(a5) and 2(b1)-2(b5), we display the schematic diagrams of the refractive index (RI) change and the corresponding photonic band gap (PBG) structure, respectively, for different
frequency detunings. From the figure one can see that the RI as well as the corresponding PBG structure (especially the Dirac cones between the first two bands) can be easily controlled by the frequency detunings.

**Figure 2 | Refractive index change and the corresponding PBG.** (a1)-(a5) RI change $\Delta n(x, y, z)$ with $\Delta_1 = -\Delta_2 = -10$ MHz (a1), $\Delta_1 = -\Delta_2 = -5$ MHz (a2), $\Delta_1 = \Delta_2 = 0$ (a3), $\Delta_1 = -\Delta_2 = 5$ MHz (a4), $\Delta_1 = -\Delta_2 = 10$ MHz (a5) and $V_0 = 60$ under the same color scale. (b1)-(b5) The corresponding PBG structures, in which the first three bands are shown. The inset in (b5) is zoomed-in plot of the first two bands.

**Photonic Floquet topological insulators.** The paraxial propagation of the probe beam in the medium can be described by the Schrödinger-like equation:

$$i \frac{\partial \psi}{\partial z} = -\frac{1}{2k_0} \nabla^2 \psi - V_0 \frac{k_0 \Delta n(x, y)}{n_0} \psi,$$

where $\psi(x, y, z)$ is the electric field envelope of the probe beam, $\nabla^2 = \partial/\partial x^2 + \partial/\partial y^2$ is the transverse Laplacian, $\Delta n(x, y, z)$ in Eq. (3) is the “effective potential” induced by the coupling fields according to Eqs. (1) and (2), $V_0$ is the potential depth, and $k_0 = 2\pi n_0/\lambda$ is the wavenumber of the probe beam, with $\lambda$ being the wavelength. In an atomic vapor, the ambient RI is $n_0 = \sqrt{1 + \text{Re}\chi_p(x, y, G = 0)}$ and the wavelength is assumed to be $\lambda = 780$ nm. In Eq. (3), the “effective potential” can then be written as

$$\Delta n(x, y) = \sqrt{1 + \text{Re}\chi_p(x, y)} - n_0 \approx \delta n_0 + \delta n_1 [2 \cos(3k_2x/2) \cos(\sqrt{3}k_2y/2) + \cos(\sqrt{3}k_2y)]$$

if only the first-order susceptibility $\chi_p^{(1)}$ is considered, where $\delta n_0 \approx \sqrt{\text{Re}\{1 + \eta\}} - n_0$ is the background uniform RI; $\delta n_1 \approx -\text{Re}\{\eta\xi\}/(2\delta n_0)$ is the coefficient for the spatially varying terms for the modulated RI with $\eta = iN\psi_0^2d_{20}/[\kappa_0(d_{10}d_{20} + 3|G_2|^2)]$ and $\xi \approx 2|G_2|^2/(d_{10}d_{20} + 3|G_2|^2)$.

We launch an obliquely incident beam on the structure and observe what happens. The initial situation is depicted in Fig. 3(a), in the momentum space; the dashed hexagon represents the first Brillouin zone. It is
clear that the obliquely incident beam can excite the zigzag edge state. In the real space, the obliquely incident beam is shown in Fig. 3(b), in which the inverted triangle is the constructed PFTI. When the probe beam propagates to a distance of $z \approx 6.3 \mu m$, the intensity distribution is numerically stimulated and exhibited in Fig. 3(c). Comparing Fig. 3(c) with Fig. 3(b), one can see that the beam moves to the left along the zigzag edge without scattering energy into the bulk of PFTI. When the beam further propagates to $z \approx 18.6 \mu m$, it moves to the bottom corner of the PFTI (Fig. 3(d)), and still there is nearly no energy scattered into the PFTI. In Figs. 3(e1)-3(e3), the frequency detunings are changed to $\Delta_1 = -\Delta_2 = -11$ MHz (Fig. 3(e1)), $\Delta_1 = -\Delta_2 = -9$ MHz (Fig. 3(e2)), and $\Delta_1 = -\Delta_2 = -8$ MHz (Fig. 3(e3)), respectively. We find that the beams in Figs. 3(e2) and 3(e3) move faster along the zigzag edge than the cases displayed in Figs. 3(c) and 3(e1), i.e., the speed is bigger if the absolute values of the frequency detunings are smaller. If the PFTI possesses a disorder, as shown in Fig. 3(f), the beam will move around it, due to the topological protection.

**Figure 3** | **Beam propagation along the edge of the PFTI.** (a) Input beam exhibited in Fourier space. The dashed hexagon is the first Brillouin zone. (b)-(d) Simulated probe beam intensity distributions when propagating at distances of $z = 0$, $z \approx 6.3 \mu m$, and $z \approx 18.6 \mu m$, respectively. The inverted triangle exhibits the modulated RI with zigzag boundaries. The parameters are $\Delta_1 = -\Delta_2 = -10$ MHz, $V_0 = 150$, $R \approx 24.8$ nm, and $\omega/(2\pi) \approx 0.8$ GHz (the period is $\sim 1.2$ nm). (e1)-(e3) Same as (c) but for $\Delta_1 = -\Delta_2 = -11$ MHz, $\Delta_1 = -\Delta_2 = -9$ MHz, and $\Delta_1 = -\Delta_2 = -8$ MHz, respectively. (f) Beam propagating to $z \approx 4.7 \mu m$, around a disorder displayed in the PFTI.

**Influence of high-order nonlinearities.** When the coupling beam intensities are high enough, the third- and fifth-order nonlinear susceptibilities ($\chi^{(3)}_p |E_2|^2$ and $\chi^{(5)}_p |E_2|^4$) must be included. Thus, the total RI change should be modified to
\[ \Delta n(x, y) \approx \delta n_0 + \delta n_1 [2 \cos(3k_2x/2) \cos(\sqrt{3}k_2y/2) + \cos(\sqrt{3}k_2y)] \\
+ \delta n_2 [2 \cos(3k_2x/2) \cos(\sqrt{3}k_2y/2) + \cos(\sqrt{3}k_2y)]^2 \\
+ \delta n_4 [2 \cos(3k_2x/2) \cos(\sqrt{3}k_2y/2) + \cos(\sqrt{3}k_2y)]^3 \]

with \( \delta n_2 \) and \( \delta n_4 \) being the higher-order nonlinearity coefficients, which connect with the Rabi frequency and frequency detunings. Here, \( \delta n_0 \approx \sqrt{1 + \Re\{1 - \tau + \tau^2\eta\}} - n_0 \),

\[ \delta n_1 \approx -\Re\{(1 - 2\tau + 3\tau^2)\eta\xi\}/(2\delta n_0), \quad \delta n_2 \approx \Re\{(1 - 3\tau + 6\tau^2)\eta\xi^2\}/(2\delta n_0) \]

and \( \delta n_3 \approx -\Re\{(1 - 4\tau + 10\tau^2)\eta\xi^3\}/(2\delta n_0) \), where \( \tau = G_2^2/(d_{10}d_{20} + G_2^2) \). With an increase in \( G_2 \), the influence of high-order nonlinear susceptibilities grows, which modifies the RI change patterns significantly. Taking \( \Delta_1 = -\Delta_2 = -10 \text{ MHz} \) as an example, \( G_2 > 19.5 \text{ MHz} \) will make the RI change be complex, so that the beam may undergo gain or absorption during propagation\(^{35}\), which may provide a new way to study the \( P\bar{T} \) symmetry in atomic ensembles\(^{36,37}\).

In Figs. 4(a1)-4(a3), we plot the real parts of the susceptibilities \( \chi_p^{(1)} \), \( \chi_p^{(3)}|E_2|^2 \) and \( \chi_p^{(5)}|E_2|^4 \), respectively, under the same color scale, with \( \Delta_1 = 0, \Delta_2 = 10 \text{ MHz} \) and \( G_2 = 10 \text{ MHz} \). It is clear that the signs of the first- and fifth-order susceptibilities are the same, while the third-order susceptibility has the opposite sign. The RI change values of \( \chi_p^{(1)} \) at the lattice sites will change when the modifications of \( \chi_p^{(3)}|E_2|^2 \) and \( \chi_p^{(5)}|E_2|^4 \) are taken into account. In Fig. 4(b) we display the total RI change pattern with all three susceptibilities considered. Therefore, the RI change values at the lattice sites can be controlled through manipulating the frequency detunings and intensities of the coupling fields. The PBG structures of the RI for \( \chi_p^{(1)}, \chi_p^{(1)} + \chi_p^{(3)}|E_2|^2 \) and \( \chi_p^{(1)} + \chi_p^{(3)}|E_2|^2 + \chi_p^{(5)}|E_2|^4 \) are shown in Figs. 4(c1)-4(c3), respectively.

**Figure 4 | Influence of higher-order nonlinearities on PFTI.** (a1)-(a3) Real parts of \( \chi_p^{(1)}, \chi_p^{(3)}|E_2|^2 \) and \( \chi_p^{(5)}|E_2|^4 \), respectively. (b) Modulated RI change patterns with \( \chi_p^{(1)}, \chi_p^{(3)}|E_2|^2 \), and \( \chi_p^{(5)}|E_2|^4 \) all considered. Parameters are: \( \Delta_1 = 0, \Delta_2 = 10 \text{ MHz} \) and \( G_2 = 10 \text{ MHz} \). (c) PBG structure for RI change with \( \chi_p^{(1)} \) (c1), \( \chi_p^{(1)} + \chi_p^{(3)}|E_2|^2 \) (c2), and \( \chi_p^{(1)} + \chi_p^{(3)}|E_2|^2 + \chi_p^{(5)}|E_2|^4 \) (c3) considered, respectively.
Discussion

As displayed in Fig. 2, the induced RI change exhibits a honeycomb profile, and the PBG structure contains 6 Dirac cones at the corners of the first Brillouin zone. However, different from those previous systems, the RI change, as well as the PBG structures, can be easily adjusted by the frequency detunings ($\Delta_1$ and $\Delta_2$), which is one of the main advantages of the current system. In Fig. 2(a2), the RI change at the honeycomb lattice sites is the smallest, as in the case of Fig. 2(a1), and there are Dirac cones in the corresponding PBG as shown in Fig. 2(b2). If $\Delta_1 = \Delta_2 = 0$, the linear susceptibility in Eq. (2) is imaginary, so that the RI change is always 0 in the transverse plane, as shown in Fig. 2(a3). Since there is no RI change when the honeycomb lattice disappears, there are no Dirac cones in the PBG, as displayed in Fig. 2(b3), in which the edges of the bands merge with each other. Note that for other cases with $\Delta_1 - \Delta_2 = 0$, the RI change will not be 0 everywhere. If $\Delta_1 < 0$, the RI change at the honeycomb lattice sites will be the smallest, while if $\Delta_1 > 0$, it will be the biggest. Therefore, even if the first two bands seemingly merge with each other, there are still Dirac cones, which is different from the case shown in Fig. 2(b3). If $\Delta_1 = -\Delta_2 = 5$ MHz, the RI change at the honeycomb lattice sites becomes the biggest, as exhibited in Fig. 2(a4), and the corresponding PBG in Fig. 2(b4) shows a big band gap between the first two bands, so that the Dirac cones disappear. If we adjust the frequency detunings to $\Delta_1 = -\Delta_2 = 10$ MHz, as shown in Figs. 2(a5) and 2(b5), the RI change is still the biggest at the lattice sites, however the first two bands become almost degenerate. If we display the first two bands exclusively, as shown in the inset in Fig. 2(b5), we find there are still 6 Dirac cones. However, the heights of the first two bands are too small to be recognized in comparison with the big band gap.

It is interesting to find out why the frequency detunings can determine whether the honeycomb RI change PBG structures have Dirac cones or not. In Figs. 2(a1) and 2(a2), the nonlinear potential or RI change at the lattice sites is deep enough to ensure the appearance of Dirac cones. With increasing frequency detunings, the nonlinear potential at or around the lattice sites becomes smaller, which cannot support the Dirac cones anymore (Figs. 2(a3) and 2(a4)). Further increasing the frequency detunings will deepen the nonlinear potential so that the Dirac cones can appear again in the PBG structure (Fig. 2(a5)). According to Eq. (3), $V_0$ is related to the potential depth, so for the same frequency detunings, the bigger $V_0$ makes the observation of Dirac cones easier.

As demonstrated previously, honeycomb lattices possess edge states when they have finite dimensions and exhibit Dirac cones in their PBG structures. If we transform coordinates by the formulas $x' = x + R \cos(\omega z)$, $y' = y + R \sin(\omega z)$, and $z' = z$ where $R$ is the radius of the helix and $\omega$ the frequency of rotation, the lattice sites of the interference honeycomb pattern will spiral along the $z$ direction (see Methods). According to the discussion on the relation between Floquet modes and helical waveguides in Ref. 9, such a spiraling honeycomb lattice from the three-beam interference can serve as a kind of PFTI formed in a multi-level atomic ensemble. When a beam propagates along the edge of PFTI, it does not scatter energy into the bulk (Figs. 3(b)-3(d)). This phenomenon can be naturally explained by the
following two reasons: (I) The zigzag edge state is excited by the obliquely incident beam, so that the confinement of the edge state prohibits the scattering of the beam during propagation. (II) The honeycomb lattice sites are helical along the $z$ direction; such a structure helps the edge state acquire an effective velocity along the propagation distance $z$, which lets the probe beam move anticlockwise along the zigzag edge.

As shown in Figs. 3(e1)-3(e3), frequency detunings may also affect the moving speed of the beams along the edge of PFTI. The reason is clear – smaller frequency detunings will lead to a shallower potential (the dips in the RI change are smaller), so that the beam can pass over successive potential barriers much easier. In Figs. 2(a5) and 2(b5), the RI also exhibits Dirac cones in the corresponding momentum space. However, we cannot construct a PFTI using the parameters from Figs. 2(a5) and 2(b5). Comparing Fig. 2(a5) with Fig. 2(a1) and considering the beam localization in Fig. 3, we can see that the beam is trapped in the regions around the dips in RI (the red regions according to the color scale), which are the real potential barriers. In Fig. 2(a5), the real potential barriers are discrete peaks, which are rather hard for the beam to overcome and obtain a transverse velocity. On the other hand, as the first two bands are almost degenerate, the edge states will be too weak to confine the beam to the edge. Various numerical simulations indicate that $\Delta_1 = -\Delta_2 > 0$ would not support the formation of PFTI. So, the appearance of Dirac cones in the momentum space is not the sufficient condition for realizing PFTI. By fixing $\Delta_1 = -\Delta_2 = -10$ MHz, we can change the intensities of the coupling fields to investigate the properties of the constructed PFTI. We find that the moving velocity of the beam increases with the increasing intensities of the coupling fields.

In Fig. 4, we also consider the influence of higher-order nonlinearities on formation of PFTI. Dirac cones clearly appear in Fig. 4(c1). However, in Fig. 4(c2) the first two bands are nearly degenerate and flat with a wide band gap between them and the third band\textsuperscript{39}, which is quite similar to the case in Fig. 2(b5). The flat bands mean that only the light with very special propagation constants can be allowed to couple among the honeycomb lattices. The reason for the appearance of the flat band is that the third-order nonlinear susceptibility makes the bands collapse\textsuperscript{40}. As a result, the PFTI cannot be formed when the third-order nonlinear susceptibility is included under such parameters. However, when the fifth-order nonlinear susceptibility is also included, we find that the Dirac cones reappear again in the first two bands, as shown in Fig. 4(c3). Therefore, the third-order nonlinear susceptibility shuts off and the fifth-order nonlinear susceptibility reopens the Dirac cones. In this manner, the higher-order nonlinear susceptibility can serve as a kind of switch, which determines the appearance of Dirac cones in the momentum space, and consequently the formation of PFTI in the system.

In summary, we have proposed a scheme to construct PFTIs in multi-level atomic ensembles. The formed PFTIs in atomic ensembles can be easily controlled and reconfigured by adjusting the frequency detunings, coupling field intensities, and high-order nonlinear susceptibilities, which shows the advantages of using atomic ensembles to study PFTI properties in comparison with other solid media. The PFTIs should also exist in other types of multi-level atomic systems. Such controllability of the spiral and the switch may establish a new platform for better understanding of the topological protection and for finding essential
photonic device applications for PFTIs.

Methods

Propagation calculation. We adopt the Split-Step Fast Fourier Transform method to perform the propagation of beams numerically.

Band structure calculation. We use the plane-wave expansion method to obtain the band structure of different interference patterns, because this method can take into account all the parameters which would affect the refractive index change. However, we do not show the energy scales in depicting photonic band gaps, since we are more interested in the Dirac cones in the band structure. The real energy scales are connected with $V_0$; different $V_0$ values (large enough to guarantee the appearance of Dirac cones) will lead to different energy scales. According to previous literatures $^{9,16,35,38}$, such band structures can also be obtained by using the tight-binding method. In addition, the full Floquet band structure and the edge band structure can also be analyzed based on this tight-binding method $^{41,42}$. In Fig. 5, we show the full band structures and the corresponding edge band structures, which are quite similar to those given in Ref. 9. The corresponding topological invariant for each band shown in Fig. 5 can be demonstrated by Chern numbers $^{43-45}$, which is given by

$$C = \frac{1}{2\pi} \int_{BZ} d^2k \nabla_k \times A(k), \quad (4)$$

where the $k$-space integral is performed over the first Brillouin zone and the Berry connection is given by

$$A(k) = i \langle u_k(r)|\nabla_k|u_k(r)\rangle = i \int d^2r u_k^*(r) \cdot [\nabla_k u_k(r)], \quad (5)$$

with $u_k(r)$ being the periodic part of the Bloch function. Since the full Floquet band structure, edge band structure, and the analysis on the topological invariants are quite similar to those given in Ref. 9, we will not explore them further here.

Figure 5 | Full Floquet band structure and the edge band structure. (a) Band structure when the
honeycomb waveguides are not helix along the propagation direction. (b) Same as (a) but the waveguides are helix along the propagation direction. (c) Corresponding to (a); edge band structure of the strained honeycomb waveguides with zigzag boundaries. (d) Same as (c), but corresponding to (b).

**Production of helical interference pattern along the propagation direction.** Although techniques of making helical waveguides in solid materials have been reported\(^{46-48}\), helical waveguides in atomic ensembles have not been reported as far. In real experimental systems, one possible method to realize the helical honeycomb pattern is to make use of the nonlinear phase shift (NPS) modulation\(^{49,50}\), by employing an additional controlling field, as shown in Fig. 1(a). The added controlling field \( E_3 \) will further split \( |+\rangle \) into \(|\pm\rangle\) with eigenfrequencies \( \ell_{|\pm\rangle} = (-\Delta_2/2 + \sqrt{\Delta_2^2/4 + |G|^2}) + (\Delta_3/2 \pm \sqrt{\Delta_3^2/4 + |G_3|^2}) \), in which \( \Delta_3 = \Delta_3 + (\Delta_2 - \sqrt{\Delta_2^2/4 + |G|^2}) \). \( \Delta_3 \) is the frequency detuning of \( E_3 \) and \( G_3 \) the Rabi frequency of \( E_3 \). The NPS can be written as \( S_{NL}(r, \phi, z) = 2k_2n^X_2 I_3 e^{-|r^2+z^2+2r l \cos(\phi-\phi^\prime)|} z/n_0 \) in the cylindrical coordinates with \( r = \sqrt{x^2+y^2} \) and \( \phi = \arctan(y/x) \), where \( l \) and \( \phi' \) are the distance and angle of the controlling field relative to the lattice site, respectively. Here, \( n^X_2 = \Re\{\chi^{(3)}X\}/(\epsilon_0 c n_0) \) is the cross-Kerr nonlinear index from the controlling field, \( \chi^{(3)}X = N\psi_1^2\psi_2^2\psi_3^2/(\hbar^3\epsilon_0 c G_p G_3^2) \), \( \rho_{10}^{(3)} \) the corresponding density-matrix element, and \( I_3 \) the intensity of the controlling field. The NPS will introduce a transverse vector \( \delta k_L(r, \phi) = \hat{r}(\partial S_{NL}/\partial r) + \hat{\phi}(\partial S_{NL}/\partial \phi)/r = k_r + k_\phi \) with \( \hat{r} \) and \( \hat{\phi} \) being the unit vectors. Specifically, the momenta can be written as \( k_r = -2S_{NL} [r + l \cos(\phi - \phi^\prime)] \hat{r} \) and \( k_\phi = -2S_{NL} l \sin(\phi - \phi^\prime) \hat{\phi} \), which drive the lattice sites to move radially (\( R \)) and azimuthally (\( \omega \)), respectively, as shown by the illustration in Fig. 6. Therefore, along the propagation direction \( z \), the honeycomb lattice will be helical, as shown by the helical curve in Fig. 6. Furthermore, the spiraling sense (clockwise or anticlockwise), the period and the radius of the helical pattern can be all adjusted by controlling the beam intensity \( I_3 \) and the nonlinear index \( n^X_2 \). In a hot atomic vapor, the propagation distance \( z \) is effectively related to the atomic density, which can be easily controlled by the temperature\(^{17,28}\). For cold cigar-shaped atomic clouds, one can adjust the length of the sample with trap potential.

**Figure 6 | Illustration of formation of helical waveguides due to NPS modulation.**

**Desired boundaries of the PFTI.** In hot atomic vapors, one can use a beam shaper to prepare the coupling fields with certain profiles. With the spatially shaped beams, the interference of the coupling fields will lead to a honeycomb lattice with well-defined boundaries. On the other hand, for cold atomic medium, the boundary can be obtained according to the boundary of the magneto-optical trap, which is wide in
comparison to the wavelength of light. If one launches a probe beam into the medium along one edge, the corresponding optical properties can be studied. According to the distributions of the zigzag edge states in momentum space, one should launch the incident beam obliquely.

References


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**Author contributions**

Y.Q.Z., M.R.B., Y.P.Z. and M.X. designed the study and analyzed the data. Y.Q.Z. and Z.K.W. perform the calculations. All authors have contributed considerably to the writing of the manuscript.

**Additional information**

**Supplementary Information** accompanies this paper at http://www.nature.com/naturecommunications

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