Continuous-variable entanglement generation using a hybrid $\mathcal{PT}$-symmetric system

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We study a system of two coupled waveguides respectively carrying optical damping and optical gain in addition to squeezing elements in one or both waveguides. Such a system is expected to generate highly entangled light fields in the two waveguides. Especially, if the light fields are strong enough, a macroscopic, continuous-variable entanglement can be achieved. We, however, show that the degree of the created entanglement is significantly affected by the quantum noises associated with the amplification and dissipation. The entanglement for the different configurations of the setup is quantified to illustrate how the quantum noises limit such entanglement as well as determine its actual time evolution patterns. In contrast to its sensitivity to the quantum noises, the degree of entanglement is irrelevant to the input light intensities, given the inputs in coherent states.

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I. INTRODUCTION

Entanglement is a purely quantum mechanical feature that distinguishes quantum systems from their classical counterparts. It has various important applications in quantum computing and quantum information science [1]. Creating the entangled quantum states in feasible ways is always a prerequisite for these applications. Nowadays, there are two main approaches to quantum information processing: (1) the “digital” approach, in which the information is encoded in quantum systems with discrete degrees of freedom (qubits or qudits) such as two polarization states of a single photon, spin-1/2 electrons, and two lowest energy levels of quantum dots; and (2) the “analog” approach, in which the quantum correlations are encoded in continuous-variable (CV) degrees of freedom such as the quadrature amplitude of a quantized harmonic oscillator, especially those of the electromagnetic field, as well as Josephson junction and Bose-Einstein condensate [2,3].

Light is a good carrier of quantum information as it interacts weakly with the environment. On the other hand, preparing, manipulating, and measuring the CV states of light is easier than the discrete photonic qubits [2]. Also, the CV quantum states are often Gaussian states, the manipulation of which is within the reach of current experimental technology. In addition, quantitative description of all properties of the Gaussian states is possible [4]. These benefits motivate one to explore the ways of generating entangled Gaussian states.

Entanglement involving light fields with high intensities is an example of the so-called macroscopic entanglement. In addition to their possible applications [2,3], macroscopic entangled light fields are especially meaningful to fundamental physics; see, e.g., the recent experimental [5,6] and theoretical studies [7–10]. In the current work, we are concerned with such light fields in a coupled optical waveguides or cavities alternately carrying gain and loss media. Recently, these systems, governed by non-Hermitian dynamics, have attracted wide experimental [11–16] and theoretical researches [17–26].

One advantage of these systems is that the light transmission patterns can be changed by simply adjusting the intensity of coupling between their components. For example, the light field amplitudes under the balanced gain and loss, where the system can manifest a parity-time ($\mathcal{PT}$) symmetry [27], will be quickly amplified once the coupling intensity is tuned below the gain-loss rate. Intuitively, this mechanism can be applied to realize entangled output light fields with high intensities by adding a squeezing element into one of the coupled components. One can also see the interest in the entanglement following the relevant non-Hermitian dynamics in many-body systems [28].

In most previous studies, the light fields in $\mathcal{PT}$-symmetric systems are treated as classical electromagnetic fields. When dealing with the entanglement of the light fields, one will encounter an indispensable factor accompanying their amplification and dissipation—the quantum noises acting as the random drives from the associated reservoirs. The quantum noises must exist as they preserve the proper commutation relations for the evolved light field operators [29]. So far, only a few recent studies [30–32] have considered the effects of the quantum noises in optical $\mathcal{PT}$-symmetric systems, including the hybrid ones with other physical elements added into the systems [33–35]. As is well known, quantum entanglement is fragile under the influence of the noises from the environment [36,37]. However, how they affect the entanglement generated by $\mathcal{PT}$-symmetric systems remains an open question. In this paper, we will clarify such effects of quantum noises by quantitatively examining their influence on the entanglement. The comparison between the generated macroscopic entanglement in the absence and presence of amplification and/or dissipation quantum noise enables one to understand their effects in $\mathcal{PT}$-symmetric systems more deeply.

The rest of the paper is organized as follows. In Sec. II, we give a rather detailed account of the concerned dynamical process. The solution to the dynamical equation is presented in terms of the quadratures of the light fields. To illustrate the effect of the added squeezing element, we discuss how it will influence the output photon numbers in Sec. III. Our main results about the CV entanglement are presented in Sec. IV, where the entanglement generated with three different configurations of the system is illustrated under
mitting light fields change from oscillatory to exponentially growing. The transition takes place at $g = J$, the exceptional point.

Unfortunately, the non-Hermitian Hamiltonian in Eq. (1) does not possess the quantum noise elements, so one cannot apply it to study the noise-sensitive phenomena such as quantum entanglement. One approach for overcoming the shortcoming is to adopt the stochastic Hamiltonian:

$$H_s = J (\hat{a} \hat{b} + \hat{a}^\dagger \hat{b}^\dagger) + i \sqrt{2 \varepsilon} [\hat{a} \hat{\xi}_a(t) - \hat{a}^\dagger \hat{\xi}_a(t)] + i \sqrt{2 \gamma} [\hat{b} \hat{\xi}_b(t) - \hat{b}^\dagger \hat{\xi}_b(t)].$$  

(2)

This Hamiltonian explicitly includes the drives from the amplification noise $\hat{\xi}_a(t)$ and dissipation noise $\hat{\xi}_b(t)$ from the respective reservoirs while the light fields are being amplified or damped (see Fig. 1). The detailed derivation of this stochastic Hamiltonian is given in Ref. [33], and a different notation for the amplification part is used in Ref. [30]. The operators of these quantum noises satisfy the following:

$$\langle \hat{\xi}_a(t) \hat{\xi}_a(t') \rangle = 0,$$

$$\langle \hat{\xi}_a(t) \hat{\xi}_b(t') \rangle = \delta (t - t'),$$

$$\langle \hat{\xi}_b(t) \hat{\xi}_b(t') \rangle = \delta (t - t').$$

(3)

where $c = a, b$, so that the evolved light field operators under dissipation and amplification will preserve the proper commutation relations. Compared to those derived with Eq. (1), the dynamical equations from the Hamiltonian in Eq. (2) contain the extra quantum noise drive terms.

To entangle the light fields, one needs to add a squeezing element into the concerned waveguide system. The action of the squeezing element with parameter $\epsilon = r \exp(i \theta)$ is described by the Hamiltonian (when it is added into waveguide $B$)

$$H_s = \frac{i}{2} [\epsilon (\hat{b}^\dagger)^2 - \epsilon^* (\hat{b})^2].$$

(4)

This Hamiltonian is based on the undepleted pump approximation for a process of second harmonic generation in a nonlinear crystal with certain symmetry (for example, LiNbO$_3$) [41]. A similar use of squeezing element was also proposed for engineering the quantum properties of other systems (see, e.g., Ref. [42]).

The dynamical evolution due to the total Hamiltonian $H = H_1 + H_s$ determine all properties of the light fields propagating in the concerned system. To solve the dynamical equations more efficiently, it is convenient to work with the quadratures of the light fields and quantum noises defined as

$$\hat{q}_c = \frac{1}{\sqrt{2}} (\hat{c} + \hat{c}^\dagger), \quad \hat{p}_c = -\frac{i}{\sqrt{2}} (\hat{c} - \hat{c}^\dagger),$$

$$\hat{Q}_c = \frac{1}{\sqrt{2}} (\hat{\xi}_c(t) + \hat{\xi}_c(t)^\dagger), \quad \hat{P}_c = -\frac{i}{\sqrt{2}} (\hat{\xi}_c(t) - \hat{\xi}_c(t)^\dagger),$$

(5)

where $c = a, b$. Then, the Heisenberg-Langevin equation for the process in Fig. 1 reads [29]

$$\frac{d}{dt} \hat{X}(t) = M \hat{X}(t) + \hat{F}(t).$$

(6)
where

\[ \dot{\mathbf{X}}(t) = \left[ \dot{\hat{q}}_a(t), \dot{\hat{p}}_a(t), \dot{\hat{q}}_b(t), \dot{\hat{p}}_b(t) \right]^T, \]  

\begin{equation}
M = \begin{pmatrix}
g & 0 & 0 & J \\
0 & g & -J & 0 \\
0 & J & -r \gamma + r \cos \theta & r \sin \theta \\
-J & 0 & r \sin \theta & -\gamma - r \cos \theta
\end{pmatrix}
\end{equation}

is the dynamic matrix, and

\[ \dot{\mathbf{F}}(t) = \left[ \sqrt{2g} \dot{\hat{q}}_a(t), -\sqrt{2g} \dot{\hat{p}}_a(t), \sqrt{2\gamma} \dot{\hat{q}}_b(t), \sqrt{2\gamma} \dot{\hat{p}}_b(t) \right]^T. \]

If there is no squeezing element \((r = 0)\) and the gain and loss rates are balanced \((g = -\gamma)\), the dynamic matrix \(M\) will be invariant after interchanging two diagonal submatrices (the operation of the parity transformation) and swapping \(g \leftrightarrow -\gamma\) (the effect of the time-reversal transformation), reflecting the \(\mathcal{PT}\) symmetry of the system. The solution to the dynamical equations takes the form

\[ \mathbf{X}(t) = e^{Mt} \mathbf{X}(0) + \int_0^t e^{M(t'-r)} \dot{\mathbf{F}}(t') \, dt'. \]  

If the input fields are coherent states, their evolved quantum states according to the above dynamical equation will be preserved in Gaussian states. All properties of such Gaussian states can be depicted by the covariance matrix (CM) \([2-4]\):

\[ V = \begin{pmatrix}
A & C \\
C^T & B
\end{pmatrix}, \]

where

\[ V_{ij} = \langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle - 2 \langle \hat{X}_i \rangle \langle \hat{X}_j \rangle. \]

The expectation values of the homogeneous part in \((10)\) are calculated with respect to the input coherent states \((\alpha, \beta)\), while those of the inhomogeneous part are found with respect to the total reservoir state \(\rho_0\) (a zero temperature for the reservoirs is assumed) via the relations in Eq. \((3)\). These CM elements can be experimentally measured \([43]\).

III. EVOLUTION OF PHOTON NUMBER AND WAVEGUIDE-MODE CORRELATION

A main purpose of the current study is to find out how the quantum noises affect the entanglement generated with the setup in Fig. 1. To see this, one can compare the values of the entanglement found as the results of the evolutions according to the Hamiltonian in Eqs. \((1)\) and \((2)\), respectively. The only difference in the Heisenberg-Langevin equation derived with the latter is an extra quantum noise drive term, \(\dot{\mathbf{F}}(t)\) in Eq. \((6)\), which consists of the components of pure random variables. By intuition, such random drives from the environment can only modify the dynamics of the system without considerably altering the evolution patterns of the measurable quantities. An example of such simple modification by insignificant quantities was found for the photon number evolution in a \(\mathcal{PT}\)-symmetric system without the squeezing element \([30]\).

A relevant question is whether the added squeezing will make a considerable difference. To answer the question, we examine how the output light fields’ photon numbers evolve according to the full dynamical equation, Eq. \((6)\). Previously, the evolved photon numbers in a \(\mathcal{PT}\)-symmetric system without squeezing have been studied for single-photon and vacuum-state inputs. Due to the amplification noise, the output photon number was found not to be zero even when the input field is in a vacuum state \([30]\). After adding the squeezing element, we find that the photon numbers can be enhanced further in addition to the effect of the gain medium at rate \(g\).

Figure 2 shows the photon numbers plotted for a setup with the squeezing element in the damping waveguide, indicating that the photon numbers will be intensified by increasing the squeezing parameter \(r\). The effect of phase factor \(\theta\) in the squeezing parameter is negligible compared to that of \(r\), and hence we will fix the value of \(\theta\) in the discussions. The contribution from the homogeneous part in Eq. \((6)\) will become much more significant with the increase of \(r\). Therefore, the relative difference between the light intensities obtained by using the Hamiltonians in Eqs. \((1)\) and \((2)\), respectively, becomes smaller than that in the previously studied situation without the squeezing; compare Fig. 2(a) with Fig. 2(d).

However, at small values of \(r\), there is a significant difference between the intensities with and without considering the quantum noise effects. As is seen from Figs. 2(a)–2(d), the corresponding photon numbers are indeed enhanced by the amplification noise.

Similar to the situation of a simple \(\mathcal{PT}\)-symmetric system without squeezing \([30]\), one sees that the quantum noises simply modify the output light intensity quantitatively, but their evolution patterns remain unchanged. As another example, we look at a nonlocal quantity, the correlation function defined as \(|\langle \hat{a}^\dagger \hat{b} \rangle / \langle \hat{a}^\dagger \rangle \langle \hat{b} \rangle - 1|\), assuming that the squeezing element is inside the damping waveguide. On the surface, such a correlation function, which evolves with time, might be similar.
to the entanglement of the light fields. Figure 3 illustrates the evolution of such a correlation function for three different values of the squeezing parameter. As expected, the correlation becomes stronger with a larger squeezing parameter. One also sees that the inclusion of quantum noise drive only slightly modifies the amplitude of the correlation without changing the time evolution pattern. In particular, it is interesting that by including the noise drives sometimes one can have a stronger correlation. Then it is natural for one to consider whether this phenomenon reflects a similar pattern for the corresponding entanglement between the output light fields.

IV. ENTANGLEMENT OF OUTPUT FIELDS

By the elements used in Fig. 1, it seems that highly entangled intense light fields can be realized with ease as the light fields keep amplified in the $PT$-symmetric broken regime. The degrees of the entanglement of Gaussian states can be measured by the logarithmic negativity, which is calculated with the CM of the two system modes [3]:

\[ E_N = \max[0, -\ln \eta], \]

(13)

where

\[ \eta = \frac{1}{\sqrt{2}} \sqrt{\Sigma - \sqrt{\Sigma^2 - 4 \det V}} \]

(14)

and

\[ \Sigma = \det A + \det B - 2 \det C. \]

(15)

Assuming the input light fields as coherent states, one finds an important property of such entanglement measured by Eq. (13). Referring to Eq. (12), one sees that the first term depends on the mode operators as well as the noise drives, whereas the second term only depends on the mode operators (the expectation values of the noise operators in it are zero). After subtracting the latter from the former, the CM elements become irrelevant to the intensity of the input fields. Consequently, the intensity of the input coherent states will be irrelevant to the generated entanglement, the degree of which is mainly influenced by the noise drives leading to the inhomogeneous part in Eq. (10).

In what follows, we will examine the output entanglement with three different configurations—the squeezing element in the damping waveguide, in the amplification waveguide, and in both waveguides.

A. Squeezing element in the damping waveguide

The first configuration we study is a coupled gain-loss waveguide system with an added squeezing element to the damping waveguide as in Fig. 1. In our calculations, we fix the phase factor $\theta$ because its variation does not affect the results. The output fields will become strongly entangled without adding the noise drive term $\hat{F}(t)$ that gives the inhomogeneous part in Eq. (10). Figures 4(a) and 4(c) show that the degree of entanglement simply increases as $r/J$. Highly entangled fields with huge photon number are thus seemingly possible by choosing the proper system parameters. One should notice that the distributions of entanglement in Figs. 4(a) and 4(c) are qualitatively the same for different damping rates, but the degrees of entanglement differ quantitatively.

The inclusion of the quantum noises associated with the amplification and dissipation will totally change the above-mentioned scenario. Now one will find that the entanglement vanishes throughout the time except when $g/J$ is very small. Without loss of generality, we demonstrate the realistic entanglement distribution at a particular time as in Figs. 4(b) and 4(d) (two different values of the damping rate $\gamma/J$ are also considered). The interplay of the quantum noises with the squeezing element under the full dynamics causes such distinct entanglement distributions from the corresponding ones without considering the noises. In contrast to the previous situation, here the distribution of the nonzero entanglement changes qualitatively with the damping rates $\gamma/J$ and the degree of entanglement changes quantitatively as well, so that the lower loss rate yields the higher entanglement.
To illustrate the effects of the quantum noises more clearly, we extend the range of $g/J$ to the negative values in Fig. 5, so that the obtained entanglement distributions also cover the situation when both waveguides contain the dissipation medium. In Fig. 5(a), where the quantum noises are not included, the values of $E_N$ distribute continuously from the positive to the negative range of $g/J$. The corresponding entanglement is shown to grow with the damping rate $\gamma/J = -g/J$, but that is impossible to occur. Such an unphysical result constitutes an evidence that quantum noises are indispensable in the study of entanglement. On the contrary, the realistic entanglement obtained under the noise effects in Fig. 5(b) takes a discontinuous transition on the boundary $g/J = 0$ because for $g/J > 0$ the amplification and dissipation noises act, but for $g/J < 0$ only the dissipation noise acts in both waveguides.

The amplification and dissipation noises act simultaneously with the squeezing that entangles the light fields, and their effects dominate over the latter when $g/J$ and $\gamma/J$ become large. The different components in the noise drive vector Eq. (9) contribute to the evolved modes via the respective elements in the dynamic matrix $M$. If the gain and loss are balanced, the impact of the amplification noise and the dissipation noise will be equal provided no squeezing element is added to the system (it can be seen from the relevant elements of the dynamic matrix $M$). However, a squeezing element can interplay with the relevant noises, and if it is placed in the damping waveguide, the influence of the dissipation noise will be enhanced. Mathematically, such interplay has more contribution from the dissipation noise operator $\hat{E}_b(t)$, which is inside the drive terms in Eqs. (9) and (10), to the CM elements. This also explains the fact that the entanglement is less influenced under the lower loss rate $\gamma/J$ since the intensity of the dissipation noise [decided by $\sqrt{\gamma}$ in Eq. (2)] becomes lower. On the other hand, regardless of how small the damping rate is, the entanglement will vanish quickly at a high gain rate even if the squeezing is large because the high gain rate leads to a significant effect of amplification noise, which can eliminate the entanglement.

Meanwhile, by comparing Fig. 3 with Fig. 4, one concludes that the existence of correlations between the two system modes does not suffice to give rise to their entanglement. The correlation under the full dynamics can be even stronger than the corresponding one without considering the noises, but the noises weaken the entanglement and, under some circumstances, they will kill the entanglement completely. The entanglement has to be determined by the relations between the CM elements involving the light field correlations, and hence its existence is much more restricted.

**B. Squeezing element in the amplifying waveguide**

If the squeezing element is inserted into the amplifying waveguide, the dynamic matrix in Eq. (8) will be changed to

$$M = \begin{pmatrix} g + r \cos \theta & r \sin \theta & 0 & J \\ r \sin \theta & g - r \cos \theta & -J & 0 \\ 0 & J & -\gamma & 0 \\ -J & 0 & 0 & -\gamma \end{pmatrix}. \tag{16}$$

The dynamical evolutions of the waveguide modes, as given by Eq. (10), will be changed accordingly. In what follows, we will examine how the entanglement between the two waveguide modes will change as the location of the squeezing element is swapped to the amplifying waveguide.

In Fig. 6(a), we illustrate the numerically simulated entanglement evolutions in the absence of the quantum noise effects. One sees that the entanglement will monotonically grow to high degrees with time. This is, however, not true in reality since the quantum noise must be considered in a real process. Figure 6(b), including the quantum noise effects, shows that the entanglement will be finally eliminated, so there is an optimum evolution time to obtain maximum entanglement. The degrees of the achieved entanglement is nonetheless higher than the situation of placing the squeezing into the damping waveguide, those illustrated in Figs. 4(b) and 4(d). Moreover, the nonzero entanglement can exist in the broader range of the parameter space; compare Fig. 7(b) with Fig. 4(b).

In this configuration, the squeezing enhances the effect of the amplification noise, but the effect of the noise accompanying the light field dissipation is kept almost invariant. One...
FIG. 7. Panels (a) and (c): the entanglement distribution (in terms of $E_N$) from a squeezing element in the amplifying waveguide without involving the noise effects. Panels (b) and (d): the corresponding entanglement distribution including the quantum noise effects. Here we choose $\theta = \pi/4$ and $J_t = 1$. In panels (a) and (b) we set $\gamma/J = 0.6$, and in panels (c) and (d) we have $\gamma/J = 0.1$.

Fig. 7(d) shows that the loss rate is reduced by 6 times to have the dissipation noise weakened accordingly. However, neither the degree of entanglement nor the range of nonzero $E_N$ is obviously changed. This is in contrast to a considerable change from Fig. 4(b) to Fig. 4(d). Such difference also implies that, given a gain rate $g/J$ that is not very large, the dissipation noise is more detrimental to the concerned entanglement.

An important feature that should be illustrated with the time evolutions is that the entanglement evolved under the full dynamics undergoes the entanglement sudden death (ESD) [37]; see Fig. 6(b). Beyond the moments of ESD when it disappears, the entanglement will stay zero forever (for

C. Squeezing elements in both waveguides

Next, we consider the setup with the squeezing elements added into both waveguides. In this situation, the dynamic matrix of the system becomes

$$M = \begin{pmatrix} g + r \cos \theta & r \sin \theta & 0 \\ r \sin \theta & g - r \cos \theta & J \\ 0 & -J & -\gamma + r \cos \theta \\ 0 & J & r \sin \theta \\ -J & 0 & -\gamma - r \cos \theta \end{pmatrix}.$$ \hfill (17)

Under the condition $g = -\gamma$, the dynamic matrix of the system exhibits a $\mathcal{P}\mathcal{T}$ symmetry. In this configuration, the degree of entanglement will be even higher than those of the two previous configurations if the quantum noises are absent; see Figs. 8(a) and 8(c). It is within the expectation since the squeezing elements act in both waveguides. Also, in the presence of the quantum noises, the degree of the entanglement is higher than those in the two other configurations as demonstrated in Figs. 8(b) and 8(d). Moreover, the range of nonzero $E_N$ becomes broader as compared to Figs. 4(b) and 4(d). Now both the amplification noise and the dissipation noise are relevant to the evolved entanglement as their effects are enhanced by the squeezing elements in both waveguides. A consequence is that bipartite entanglement of Gaussian states, the logarithmic negativity calculated with Eqs. (13)–(15) is a well-defined quantity). By the scenario in Fig. 6, one concludes that the quantum noises control the actual time evolution pattern of the entanglement.

V. CONCLUSION

We have studied how quantum noises influence the CV entanglement generated with a hybrid $\mathcal{P}\mathcal{T}$-symmetric setup. By intuition, the existing quantum noises associated with amplification and dissipation would only modify the

FIG. 8. Distribution of entanglement (in terms of $E_N$) generated by placing identical squeezing elements into both waveguides. We set $\theta = \pi/4$, and $J_t = 1$. In panels (a) and (c), the entanglement values are calculated without considering quantum noises, and in panels (b) and (d), the entanglement values are found under the full dynamics. In panels (a) and (b), we set $\gamma/J = 0.6$, and in panels (c) and (d), we have $\gamma/J = 0.1$. 

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bipartite entanglement of Gaussian states, the logarithmic negativity calculated with Eqs. (13)–(15) is a well-defined quantity). By the scenario in Fig. 6, one concludes that the quantum noises control the actual time evolution pattern of the entanglement.

V. CONCLUSION

We have studied how quantum noises influence the CV entanglement generated with a hybrid $\mathcal{P}\mathcal{T}$-symmetric setup. By intuition, the existing quantum noises associated with amplification and dissipation would only modify the
entanglement generated by the system slightly. Then, according to the prediction by the non-Hermitian Hamiltonian in Eq. (1), highly entangled, high-intensity light fields can be readily created by such a setup, especially by the system operating in the $PT$-symmetry-broken regime ($g/J > 1$) where the light fields can be amplified and entangled at the same time. As a matter of fact, however, the quantum noises can completely eliminate the entanglement, rendering its evolution totally different from those of photon numbers and field-mode correlations. The results obtained by the full dynamics indicate that certain amounts of the entanglement can still be achieved, though they are weaker than those predicted without considering the quantum noises. In particular, placing the squeezing element inside the waveguide amplifying the propagating light field enables one to realize light fields with considerable CV entanglement, given the gain rate $g/J$ that is not too large. The possible experimental realization of such systems relies on finding a material or a method to purely amplify and squeeze the input light simultaneously. The importance of studying this model setup is to clarify that quantum noises must be considered in $PT$-symmetric optical systems for engineering the quantum properties of light fields.

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